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Hamiltonian Chromatic Number of Graphs

Dr. B. Ramireddy[1], U. Mohan Chand[2], A.Sri Krishna Chaitanya[3], Dr. B.R. Srinivas[4]
1) Professor & H.O.D, Hindu College, Guntur, (A.P.) INDIA.
2) Associate Professor of Mathematics & H.O.D, Rise Krishna Sai Prakasam Group of Institutions, Ongole, (A.P) INDIA.
3) Associate Professor of Mathematics & H.O.D, Chebrolu Engineering College, Chebrolu, Guntur Dist. (A.P)
4) Professor of Mathematics, St. Mary's Group of Institutions, Chebrolu, Guntur Dist. (A.P) INDIA.

Abstract
This paper studies the Hamiltonian coloring and Hamiltonian chromatic number for different graphs .the main results are1.For any integer n greater than or equal to three, Hamiltonian chromatic number of C_n is equal to n-2. 2. G is a graph obtained by adding a pendant edge to Hamiltonian graph H, and then Hamiltonian chromatic number of G is equal to n-1. 3. For every connected graph G of order n greater than or equal to 2, Hamiltonian chromatic number of G is not more than one increment of square of (n-2).
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Key words: Chromatic number, Hamiltonian coloring, Hamiltonian chromatic number, pendant edge, spanning connected graph.

I. Introduction

Generally in a (d – 1) radio coloring of a connected graph G of diameter d, the colors assigned to adjacent vertices must differ by at least d-1, the colors assigned to two vertices whose distance is 2 must differ by at least d-2, and so on up to antipodal vertices, whose colors are permitted to be the same. For this reason, (d-1) radio colorings are also referred to as antipodal colorings.

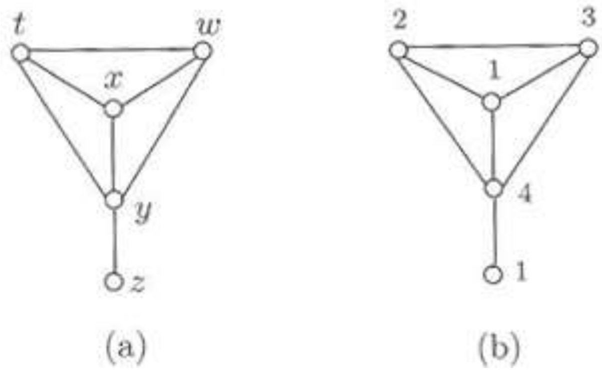
In the case of an antipodal coloring of the path P_n of order $n \geq 2$, only the two end-vertices are permitted to be colored the same. If u and v are distinct vertices of P_n and d (u, v) = i, then $|c(u)-c(v)| \geq n - 1 - i$. Since P_n is a tree, not only is i the length of a shortest u – u path in P_n, it is the length of the only u – v path in P_n. In particular, is the length of a longest u – v path?

The detour distance D (u, v) between two vertices u and v in a connected graph G is defined as the length of a longest u – v path in G. Hence the length of a longest u – v path in P_n is D (u, v) = d (u, v). Therefore, in the case of the path P_n, an antipodal coloring of P_n can also be defined as a vertex coloring c that satisfies.

$\mathbf{D(u,v) + |c(v)| \geq n - 1}$, for every two distinct vertices u and v of P_n.

§1.1 Definition: Vertex coloring c that satisfy were extended from paths of order n to arbitrary connected graphs of order n by Gary Chartrand, Ladislav Nebesky, and Ping Zhang .A **Hamiltonian coloring** of a connected graph G of order n is a vertex coloring c such that, $\mathbf{D(u,v) + |c(v)| \geq n - 1}$, for every tow distinct vertices u and v of G. the largest color assigned to a vertex of G by c is called the **value** of c and is denoted by hc(c). The **Hamiltonian chromatic number** hc (G) is the smallest value among all Hamiltonian colorings of G.

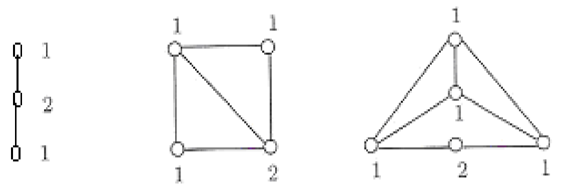
EX: Figure.1 (a) shows a graph H of order 5. A vertex coloring c of H is shown in Figure.1 (b). Since $D(u, v) + |c(u) - c(v)| \geq 4$ for every two distinct vertices u and u of H, it follows that c is a Hamiltonian coloring and so hc(c) =4. Hence hc (H) $\geq$ 4. Because no two of the vertices t, w, x, and y are connected by a Hamiltonian path, these must be assigned distinct colors and so hc (H) $\geq$ 4. Thus hc (H) = 4.



1. A graph with Hamiltonian chromatic number 4

If a connected graph G of order n has Hamiltonian chromatic number 1, then D(u,v) = n – 1 for every two distinct vertices u and v of G and consequently G is Hamiltonian-connected, that is, every two vertices of G are connected by a

Hamiltonian pat. Indeed, hc (G) = 1 if and only if G is Hamiltonian – connected. Therefore, the Hamiltonian Chromatic Number of a connected graph G can be considered as a measure of how close G is to being Hamiltonian connected. That is the closer hc (G) is to 1, the closer G is to being Hamiltonian connected. The three graphs H_1, H2 and H_3 shown in below figure 2 are all close (in this sense) to being Hamiltonian-connected since hc (H_i) = 2 for i = 1, 2, 3.



H_1 H_2
H_3
2. Three graphs with Hamiltonian chromatic number 2

II. Theorem: For every integer n $\geq$ 3, hc ($K_{1,n-1}$) = $(n-2)^2 + 1$

Proof: Since hc ($K_{1,2}$) = 2 (See H_1 in Figure 2) we may assume that $n \geq 4$.
Let G = $K_{1,n-1}$ where V(G) = $\{v_1, v_2 \ldots . v_n\}$ and v_n is the central vertex. Define the coloring c of G by c (v_n) = 1 and C (v_i) = (n-1) + (i-1)(n-3) for $1 \leq i \leq n$ - 1.Then c is a Hamiltonian coloring of G and
hc (G) $\leq$ hc (c) = c (v_{n-1}) = (n-1) + (n-2) (n-3) = $(n-2)^2$ + 1.It remains to show that hc (G) $\geq (n-2)^2 + 1$.
Let c be a Hamiltonian coloring of G such that hc(c) = hc (G). Because G contains no Hamiltonian path, c assigns distinct colors to the vertices of G. We may assume that C (v_1) < c (v_2) < … < c (v_{n-1}). We now consider three cases, depending on the color assigned to the central vertex v_n.

Case 1.
c (v_n) = 1.
Since
D (v_1, v_n) = 1 and D (v_i, v_{i+1}) = 2 for $1 \leq i \leq n-2$.
It follows that
C (v_{n-1}) $\geq$ 1 + (n-2) + (n-2) (n-3) = $(n-2)^2 + 1$
And so
<u>hc</u> (G) = hc(c) = c (v_{n-1}) $\geq (n-2)^2 + 1$.

<u>**Case 2**</u>.
C (v_n) = hc(c)
Thus, in this case,
1 = c (v_1) < c (v_2) < … < c (v_{n-1}) < c (v_n)
Hence
C (v_n) $\geq$ 1 + (n-2)(n-3) + (n-2) = $(n-2)^2 + 1$
And so
hc (G) = hc (v) = c(v_n) $\geq (n-2)^2$ +1.

<u>**Case 3.**</u>
C (v_j) < c (v_n) < c (v_{j+1}) for some integer j with $1 \leq j \leq n - 2$.
Thus c(v_1) = 1 and c(v_{n-1}) = hc(c).
in this case
C (V_j) $\geq$ 1 + (j-1) (n-3),
C (vn) $\geq$ c (v_j) + (n-2)
C (v_{j+1}) $\geq$ c (v_n) + (n-2) and
C (v_{n-1}) $\geq$ c (v_{j+1}) + [(n-1) – (j+1)] (n-3).
Therefore,
C (v_{n-1}) $\geq$ 1 + (j-1)(n-3) + 2(n-2) + (n-j-2)(n-3)
= (2n-3) + $(n-3)^2 = (n-2)^2 + 2 > (n-2)^2 + 1$.
And so hc (G) = hc(c) = c (v_{n-1}) > $(n-2)^{2+}$ 1.
Hence in any case,
hc (G) $\geq (n-2)^2$ + 1 and so hc (G) = $(n-2)^2$ + 1.

III. Theorem: For every integer n $\geq$ 3, hc (Cn) = n-2.

Proof.
Since we noted that hc (C_n) = n-2 for n = 3, 4, 5. We may assume that $n \geq 6$. Let C_n = (v_1, $v_2 \ldots v_n$, v_1). Because the vertex coloring c of C_n defined by c (v_1) = c(v_2) = 1, c(v_{n-1}) = c(v_n) = n-2 and c(v_i) = i-1 for $3 \leq i \leq n-2$ is a Hamiltonian coloring, it follows that hc(C_n) $\leq$ n-2.Assume, to the contrary , that hc (C_n) < n-2 for some integer $n \geq 6$. Then there exists a Hamiltonian (n-3) coloring c of C_n. We consider two cases, according to whether n is odd or n is even.

Case 1.
<u>**n is odd:**</u> Then n = 2k + 1 for some integer $k \geq 3$. Hence there exists a Hamiltonian (2k-2) coloring c of C_n. Let,
A = {1, 2… k-1} and B = {k, k+1…2k-2}
For every vertex u of C_n, there are two vertices v of C_n such that D (u,v) is minimum (and d(u,v) is maximum), namely D(u,v) = d(u,v) + 1 = k+1. For u = vi, these two vertices v are v_{i+k} and v_{i+k+1} (where the addition in i + k and i + k + 1 is performed modulo n).Since c is a Hamiltonian coloring.
D (u, v) + |c (u) – c (v)| $\geq$ n =l =2k.BecauseD (u, v) = k + 1, it follows that
|c(u) – c(v)| $\geq$ k – 1.
Therefore, if c (u) $\in$ A, then the colors of these two vertices v with this property must belong to B. In particular, if c (v_i) $\in$ A, then (v_{i+k}) $\in$ B. Suppose that there are a vertices of C_n whose colors belong to A and b vertices of C_n whose colors belong to B. Then b$\geq$a However, if c (v_i) $\in$ B, then c ($v_{i+k)}$ belongs to a implying that a $\geq$ b and so. a=b. since a + b = n and n is odd, this is impossible.

Case 2.
<u>**n is even:**</u> Then n = 2k for some integer $k \geq 3$. Hence there exists a Hamiltonian (2k-3) - coloring c of C_n. For every vertex u of C_n, there is a unique vertex v of C_n for which D (u, v) is minimum (and d (u, v) is maximum), namely, d (u, v) = k. For u = v_i, this

vertex v is v_{i+k} (where the addition in i + k is performed modulo n).

Since c is a Hamiltonian coloring, D (u,v) + |c(u) – c(v)| $\geq$ n – 1 = 2k – 1. Because D (u, v) = k, it follows that |c (u) – c (v)| $\geq$ k – 1. This implies, however, that if
c (u) = k-1, then there is no color that can be assigned to u to satisfy this requirement. Hence no vertex of C_n can be assigned the color k- 1 by c.
Let, A = {1, 2… k-2} and B = {k, k+1…2k-3}.
Thus |A| = |B| = k – 2. If c (v_i) $\in$ A, then c (v_{i+k}) $\in$ B. Also, if c (v_i) $\in$ B, then c (v_{i+k}) $\in$ A. Hence there are k vertices of C_n assigned colors from B.Consider two adjacent vertices of C_n, one of which is assigned a color from A and the other is assigned a color from B. We may assume that c (v_1) $\in$ A and c (v_2) $\in$ B. Then c (v_{k+1}) $\in$ B. Since D (v_2,v_{k+1}) = k+1, it follows that |c (v_2) – c(v_{k+1}) $\geq$ k – 2. Because c (v_2), c (v_{k+1}) $\in$ B, this implies that one of c (v_2) and c (v_{k+1}) is at least 2k-2. This is a contradiction.

§ 3.1Proposition: If H is a spanning connected sub graph of a graph G, then hc (G) $\leq$ hc (H)
Proof.
Suppose that the order of H is n. Let c be a Hamiltonian coloring of H such that hc(c) hc (H). Then D_H (u,v) + |c(u) – c(v)| $\geq$ n -1 for every two distinct vertices u and v of H. since D_G (u,v) $\geq$ D_H (u,v) for every two distinct vertices u and v of H, it follows that D_G (u,v) + |c(u) – c(v)| $\geq$ n – 1 and so c is a Hamiltonian coloring of G as well. Hence hc (G) $\leq$ hc (c) = hc (H).

§ 3.2 Proposition: Let H be a Hamiltonian graph of order n – 1 $\geq$ 3. If G is a graph obtained by adding a pendant edge to H, then he (G) = n – 1.
Proof. Suppose that C = (v_1,v_2…v_{n-1},v_1) is a Hamiltonian cycle of H and v_1v_n is the pendant edge of G. Let c be a Hamiltonian coloring of G. Since D_G (u, v) $\leq$ n-2 for every two distinct vertices u and v of C, no two vertices of C can be assigned the same color by c. Consequently, hc (c) > n – 1 and so hc (G) $\geq$ n – 1.
Now define a coloring c\` of G by

$$C^1(v_i) = \begin{cases} i & \text{if } 1 < i < n-1 \\ n-1 & \text{if } i = n. \end{cases}$$

We claim that c\` is a Hamiltonian coloring of G. First let v_j and v_k be two vertices of C where 1 $\leq$ j < k $\leq$ n – 1. The |c^1+ (v_j) – c^1 (v_k)| = k – j and
D (v_j,v_k) = max {k-j, (n-1) – (k-j)}.
In either case, D (v_j,v_k) $\geq$ n-1 + j – k and so
D (v_j,v_k) + |c^1 (v_j) – c^1 (v_k) | $\geq$ n-1.
For 1$\leq$ j $\leq$n-1, |c^1 (v_j) – c^1)v_n)| = n-1-j, while
D (v_j,v_n) $\geq$ max {j, n-j+1}
And so, D (v_j, v_n) $\geq$ j.
Therefore,
D (v_j, v_n) + |c^1 (v_j) – c^1 (v_n)| $\geq$ n-1.
Hence, as claimed, c’ is a Hamiltonian coloring of G and so hc (G) $\leq$ hc (c’) =c^1 (v_n) = n-1.

IV. Theorem: for every connected graph G of order n ≥ 2, hc (G) ≤ $(n-2)^2$ + 1.

Proof. First, if G contains a vertex of degree n-1, then G contains the star $K_{1,n-1}$ as a spanning sub graph. Since hc ($K_{1,n-1}$) = $(n-2)^2$ + 1 it follows by proposition 1 that hc(G) $\leq$ $(n-2)^2$ + 1. Hence we may assume that G contains a spanning tree T that is not a star and so its complement T contains a Hamiltonian path P = (v_1,v_2….v_n). Thus v_i v_{i+1} $\notin$ E (T) for 1 $\leq$ i$\leq$ n-1 and so D_T (v_i,v_{i+I}) $\geq$2. Define a vertex coloring c of T by
C (v_i) = (n-2) + (i-2) (n-3) for 1 $\leq$ i$\leq$ n.
Hence
hc (c) = c (v_n) = (n-2) + (n-2) (n-3) = $(n-2)^2$
Therefore, for integers i and j with 1 $\leq$ i< j $\leq$ n,

|c (v_i) – c (v_j)| = (j-i) (n-3).
If j = i+ 1, then

D (v_i,v_j) + (c (v_i) – c (v_j)| $\geq$ 1 + 2(n-3) = 2n-5 $\geq$ n-1.
Thus c is a Hamiltonian coloring of T. therefore,

hc (G) $\leq$ hc (T) $\leq$ hc(c) = c (v_n) = $(n-2)^2$ < $(n-2)^2$ + 1,
Which completes the proof

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